FINDING THE SHAPE OF A BODY FROM A GIVEN IMPACT PRESSURE

(NAKHOZHDENIE FORMY TELA PO ZADANNOMU IMPUL'SIVNOMU DAVLENIIU PRI UDARE)

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It is required to determine the contour of a part of a body in contact with incompressible liquid from a given impact pressure. Let us examine the two dimensional case of a liquid with infinite depth. The abscissa is assumed to be along the surface of the liquid and the ordinate is pointed vertically downward. Let the body intersect the abscissa axis at $x = \pm 1$. The velocity potential ϕ is related to the impulse pressure p_t by $\phi\rho = - p_t$ where ρ = density of the liquid. Therefore, for any given pressure along the contour L there is a corresponding potential ϕ.

Let the projections of velocities upon the axes of coordinates by u_0 and v_0, and the angular velocity ω_0, then the stream function ψ along L will be [1]

$$\psi_L = u_0 y - v_0 x - \frac{1}{2}\omega_0 (x^2 + y^2) \tag{1}$$

In formulating the reverse problem, let us assume that u_0, v_0 and ω_0 are given.

The choice of given ϕ along L depends upon the character of the motion of the body in the interval of time immediately after the shock.

First case. Let $\omega_0 = 0$, $u_0 = 0$. Let us assume that ϕ is given as a function of x:

$$\varphi_L = v_0 \Omega(x), \quad (-1 \leqslant x \leqslant 1), \qquad (\Omega(\pm 1) = 0) \tag{2}$$

From (1) it follows that $\psi_L = - v_0 x$. Therefore, to the region B_z filled with the flow, in the plane w, there is a corresponding region B_w, limited by the vertical segment $\phi = 0$, $|\psi| \leqslant v_0$, representing the free surface of the liquid and by the curve L_w conformable to the unknown contour, the parametric equation of which is $\phi = v_0\Omega(x)$, $\psi = - v_0 x$. Let us construct the function $w = w(\zeta)$, which maps the lower semi-plane Im $\zeta < 0$ upon the region B_w so that the points $\zeta = \pm 1$ are transformed into $w = \pm iv_0$, and $\zeta = \infty$ into $w = 0$. Now, the function $\Phi(\zeta) = z[w(\zeta)]$ is easily determined. Indeed, for the $\Phi(\zeta)$ we have the following conditions for the contour:

$$\operatorname{Re}\Phi(\xi) = -\frac{\psi}{v_0}\quad(-1<\xi<1),\qquad \operatorname{Im}\Phi(\xi)=0\qquad(\xi<-1,\ \xi>1)\tag{3}$$

The function ψ can be expressed through ξ from the equation $\phi = i\psi = w(\zeta)$.

Using the Keldysh-Sedov formula and noting that $\Phi(\zeta) = O(\zeta)$ when $\zeta \to \pm\infty$, we obtain:

$$z = \Phi(\zeta) = \frac{\sqrt{\zeta^2-1}}{\pi i}\int_{-1}^{1}\frac{\psi(\tau)\,d\tau}{v_0\sqrt{\tau^2-1}\,(\tau-\zeta)} + C\sqrt{\zeta^2-1}\tag{4}$$

where C is an arbitrary real constant.

Example. If $\phi_L = -v_0\cdot\sqrt{1-x^2}$, the contours form a family of ellipses with the common semi axis $-1 \leqslant x \leqslant 1$, $y = 0$ and with a variable other semi axis. This family includes the unit circle and also a flat plate.

Second Case. Let $\omega_0 = 0$ but $u_0 \neq 0$. Let us assume that $u_0^2 + v_0^2 = 1$, and introduce a new system of coordinates

$$x^* = v_0x - u_0y,\qquad y^* = u_0x + v_0y$$

The potential ϕ along the unknown contour L, as a function of x^* is given by

$$\varphi_L = \Omega(x^*)\tag{5}$$

For ψ_L in this case we have the condition $\psi_L = -x^*$. As in the first case, for the function $z^* = x^* + iy^* = \Phi(\zeta)$ which maps Im $\zeta < 0$ upon the region filled by streamflow, we obtain the following boundary condition:

$$x^* = -\psi(\xi)\,(-1<\xi<1),\qquad x^*u_0 - y^*v_0 = 0\qquad(\xi<-1,\ \xi>1)$$

This boundary value problem is a particular case of Hilbert's solution with discontinuous coefficients [2]

$$\operatorname{Re}\{[a(\xi)+ib(\xi)]\,\Phi(\xi)\} = g(\xi)\tag{6}$$

where

$$g(\xi) = -\psi(\xi),\qquad a(\xi)=1,\qquad b(\xi)=0\qquad(-1<\xi<1)$$
$$a(\xi)=u_0,\qquad b(\xi)=v_0,\qquad g(\xi)=0\qquad(\xi<-1,\ \xi>1)$$

This can be reduced to the problem of linear conjugate

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + h(\xi) \tag{7}$$

$$h(\xi) = \frac{2g}{a+ib} = \begin{cases} -2\psi(\xi) & (-1<\xi<1) \\ 0 & (\xi<-1,\ \xi>1) \end{cases}$$

$$G(\xi) = -\frac{a+ib}{a-ib} = \begin{cases} -1 & (-1<\xi<1) \\ -e^{2i\alpha} & (\xi<-1,\ \xi>1) \end{cases} \qquad (\alpha = \arg(u_0+iv_0))$$

If the functions $\Phi^+(\zeta)$ and $\Phi^-(\zeta)$ are regular respectively within the upper and the lower semiplanes, satisfy the boundary condition (7) and are related by

$$\overline{\Phi^+}(\bar{\zeta}) = \Phi^-(\zeta)$$

then the function $\Phi^-(\zeta)$ will be the solution of the boundary value problem (6). The solution of problem (7) can be obtained by utilizing the general theory of boundary value problems with discontinuous coefficients [2,3]. The solution bounded at the points $\xi = \pm 1$, of the order $O(\zeta)$ when $\zeta \to \pm\infty$ and satisfying condition $\Phi^+(\zeta) = \Phi^-(\zeta)$, will be:

$$\Phi^-(\zeta) = \Phi(\zeta) = (\zeta+1)^{\frac{\alpha}{\pi}}(\zeta-1)^{1-\frac{\alpha}{\pi}}[ie^{i\alpha}C + \Phi_0^-(\zeta)]$$

$$\Phi_0(\zeta) = -\frac{1}{2\pi i}\int_{-1}^{1}\frac{\psi(\tau)}{\tau-\zeta}(\tau+1)^{-\frac{\alpha}{\pi}}(\tau-1)^{-1+\frac{\alpha}{\pi}}d\tau \tag{8}$$

(C is real)*

Therefore, for a given distribution of impulse pressure there is a corresponding family of monoparametric contours. Formula (8) gives the solution for the case when $u_0 = w_0 = 0$, which was examined above.

Third Case. Let $\omega_0 \neq 0$, and assume that potential ϕ of the unknown contour is a function of distance to the instantaneous center of rotation of the body

$$\varphi_L = \Omega\left(\sqrt{(x+v_0/\omega_0)^2 + (y-u_0/\omega_0)^2}\right) \tag{9}$$

The condition (9) in conjunction with (1) determines in the plane w, the image of the curve of the contour L_z. (It is assumed that all the points of the contour are at various distances from $z_0 = -v_0/w_0 + iu_0/w_0$.

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* The branch $(\zeta+1)^{\alpha/\pi}(\zeta-1)^{1-\alpha/\pi}$ of this many values function is regular in the plane of the cuts along abscissa axis $-\infty < \xi < -1$ and $1 < \xi < \infty$ and real on the lower boundary of the right cut.

or the contour is composed of segments satisfying this condition.) As before, let us map the region B_w upon the lower semi-plane Im $\zeta < 0$ and examine the function $F(\zeta) = \Phi(\zeta) - z_0$ where $\Phi(\zeta) = z[w(\zeta)]$. The boundary condition for $F(\zeta)$ is: $(-1 < \xi < 1)$

$$|F(\xi)| = |z - z_0| = \sqrt{\left(x + \frac{v_0}{\omega_0}\right)^2 + \left(y - \frac{u_0}{\omega_0}\right)^2} = \sqrt{-\frac{2}{\omega_0}\psi(\xi) + \frac{u_0^2 + v_0^2}{\omega_0}}$$

$$\operatorname{Im} F(\xi) = -\frac{u_0}{\omega_0} \qquad (\xi < -1,\ \xi > 1) \tag{10}$$

Therefore, we obtain a nonlinear boundary value problem for the function $F(\zeta)$, regular in the lower half-plane and of the order of $O(\zeta)$ when $\zeta \to \pm\infty$. In the case when $u_0 = 0$, i.e. z_0 is located on the abscissa axis, the problem is easily linearized and solved. Let us assume that z_0 lies between the points $\xi = -1$ and $\xi = 1$. Let

$$F_1(\zeta) = \ln F(\zeta)$$

Then

$$\operatorname{Im} F_1(\xi) = 0 \quad \text{для } \xi < -1, \qquad \operatorname{Im} F_1(\xi) = \pi \quad \text{для } \xi > 1$$

$$\operatorname{Re} F_1(\xi) = \sqrt{-\frac{2}{\omega_0}\psi(\xi) + \frac{v_0^2}{\omega_0^2}} \qquad \text{для } -1 < \xi < 1,$$

$$F_1(\zeta) = O(\ln\zeta) \quad \text{при } z \to \infty$$

The solution of this boundary value problem is the following function

$$F_1(\zeta) = \ln(\zeta + \sqrt{\zeta^2 - 1} + \frac{\sqrt{\zeta^2 - 1}}{\pi i} \int_{-1}^{1} \frac{C(\tau)}{\sqrt{\tau^2 - 1}} \frac{d\tau}{\tau - \zeta}$$

where

$$C(\tau) = \sqrt{-\frac{2}{\omega_0}\psi(\xi) + \frac{v_0^2}{\omega_0^2}}$$

and for $z(\zeta)$ we obtain

$$z(\zeta) = z_0 + (\zeta + \sqrt{\zeta^2 - 1}) \exp \frac{\sqrt{\zeta^2 - 1}}{\pi i} \int_{-1}^{1} \frac{C(\tau)}{\sqrt{\tau^2 - 1}} \frac{d\tau}{\tau - \zeta} \tag{11}$$

Let us note that the above solution does not hold when $w_0 = 0$, since in this case, the method of assigning ϕ along the contour must be altered. If $u_0 \neq 0$, the boundary value problem (10) reduces to a nonlinear singular integral equation. Let $F_2(\zeta) = iF(\zeta)$, then

$$|F_2(\xi)| = h(\xi) \quad \text{for } -1 < \xi < 1, \qquad \operatorname{Re} F_2(\xi) = u_0/\omega_0 \quad \text{for } \xi < -1 \text{ и } \xi > 1$$

Putting $\theta(\xi) = \operatorname{ang} F_2(\xi)$, then utilizing the Schwartz integral, we obtain for $-1 < \xi < 1$ the equation:

$$C(\xi)\sin\theta(\xi) = \frac{1}{\pi}\int_{-1}^{1}\frac{C(t)\cos\theta(t)}{t-\xi}\,\frac{1+t\xi}{1+t^2}\,dt - \frac{u_0}{\pi\omega_0}\ln\frac{1-\xi}{1+\xi}$$

The results obtained when $\omega_0 = 0$ can be applied to the cases of the impact of several bodies, as well as for a liquid of a finite depth, provided that the boundary is made of straight line segments.

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